Computational Journal of Mathematical and Statistical Sciences
3(2), 389–417
DOI:10.21608/CJMSS.2024.297407.1056
https://cjmss.journals.ekb.eg/

On Fitting Renewable Energy Sources Data: Using a New Trigonometric Statistical Model

Mahmoud Elsehetry[1]*, Ahmed W. Shawki[2], Mohamed G. Khalil[3], and Tamer S. Helal[4]

[1] Department of Basic Sciences, Higher Institute for Administrative Sciences and Foreign Trade, Cairo, Egypt; ma_sehetry@hotmail.com

[2] Central Agency for Public Mobilization & Statistics, Cairo, Egyp; ahmed23484@hotmail.com

[3] Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt

[4] Department of Basic Sciences, Al Madina Higher Institute of Management and Technology, Giza, Egypt; tamerhelal2006@yahoo.com

* **Correspondence:** ma_sehetry@hotmail.com

Abstract: The goal of this work is to create an innovative heavy-tailed distribution known as the arctan-Kumaraswamy exponential (ATKE) distribution. The Kumaraswamy exponential distribution and the arctan-X family of distributions were combined to create the ATKE distribution. The ATKE distribution is adaptable and capable of modeling a range of hazard rate shapes when compablack to the conventional Kumaraswamy exponential distribution. Different asymmetric and unimodal forms are seen in the densities. The many types of decreasing, rising, increasing-constant, and reversed j-shaped shapes are depicted by the hazard rate functions. The created model is evaluated from a statistical viewpoint. Various metrics of uncertainty are calculated. Six commonly applied statistical techniques are employed, in the field of research, to estimate the parameters of the distribution. To illustrate the effectiveness of the maximum likelihood, Cramer-von Mises, least squares, Anderson-Darling (AD), weighted least squares, and the right-tail AD estimators of the ATKE distribution parameters, we conducted an extensive simulation analysis. Additionally, the adaptability of the provided model was examined using a dataset of renewable energy sources, demonstrating that, in comparison to some other competing models that contain two, three and four parameters, the suggested model could potentially utilize to fit these data.

Keywords: Arctan-G family; Kumaraswamy exponential distribution; right-tail Anderson-Darling; Entropy; Simulation.
Mathematics Subject Classification: 60E15, 62H99
Received: 13 June 2024; **Revised:** 30 June 2024; **Accepted:** 1 July 2024; **Published:** 2 July 2024.

1. Introduction and Motivation

There are situations in which the process of fitting the current distributions to a collection of data results in an inadequate fit. Many statisticians strive to generalize distributions to solve this problem and achieve a fit appropriate for a particular data set. A great deal of research has been done to create distributions that have more flexible and desirable features so that real-world data sets with different failure rate functions and densities may be properly modeled. To improve data modeling capabilities, researchers are now working on developing new hybrid distributions that generalize already-existing ones. A family of distributions and a baseline distribution are combined to create these hybrid distributions. Numerous statisticians have worked towards creating families that include generalized odd Burr III-G [1], truncated Muth-G [2], Marshall-Olkin odd Burr III-G [3], generalized truncated Fréchet-G [4], odd inverse power generalized Weibull-G [5], generalized inverted Kumaraswamy-G [6], compounded Bell-G [7], DUS transformation-G [8], weighted exponentiated-G [9], generalized DUS transformation-G [10], a new extended cosine-G [11], type II exponentiated half logistic-G [12], odd Nadarajah-Haghighi-G [13], ratio exponentiated general-G [14], unit exponentiated half logistic power series-G [15], power inverted Topp-Leone-G [16], power Lindely-G [17], inverse Weibull-G [18], Poisson-G [19], weighted-G [20], Odd Chen-G [21], Arctan-X-G [22], Arcsine-X-G [23], Zubair-G [24], Exponential T-X-G [25], new Kumaraswamy-G [26], Teissier-G [27], Amoroso-G [28], cosine Fréchet [29], sine generalized linear exponential [30], sine Kumaraswamy-G [31], new hyperbolic Sine-Rayleigh [32], arcsin inverse Weibull [33] among others.

The majority of the aforementioned generalization techniques has some drawbacks, including increasing the number of parameters in the probability model increases its flexibility; nevertheless, this frequently leads to reparameterization issues. As a result, examination of the model parameters becomes more challenging as the number of parameters increases. In addition to, the tractability of the cumulative distribution function (CDF) is decreased by many extending approaches, which increases the difficulty of manually calculating statistical characteristics. To overcome these problems, some researchers claim decided to use the trigonometric functions to seek for new distributions to seek for novel distributions [34, 35, 36, 37, 38]. As a result, several trigonometric-G families were proposed by several authors, our interest here with the arctan-X (AT-X) family presented by [39]. The following are the cumulative distribution function (CDF) and probability density function (PDF) of the AT-X family of distributions:

$$F(z;\Theta) = \frac{4}{\pi}\arctan\left(K(z;\Theta)\right), \quad z \in \mathbb{R}, \tag{1.1}$$

and

$$f(z;\Theta) = \frac{4k(z;\Theta)}{\pi\left[1 + (K(z;\Theta))^2\right]}, \quad z \in .\mathbb{R}. \tag{1.2}$$

where, $k(z;\Theta)$ and $K(z;\Theta)$ are the PDF and CDF of the base-line distribution while Θ is the set of parameters. As mentioned by [39], one of the many advantages of the AT-X family is that its highly expressive PDF and closed CDF form. On the other hand, the exponential distribution is maybe the statistical distribution that is used the most frequently to solve reliability-related issues. Scholars have been creating different versions and modifications of the exponential distribution for a long time. Recently, the KE distribution, an extension of the exponential distribution, was created by [40], as a model to address issues in survival analysis and environmental research. The PDF and CDF, with set of parameters $\Theta = (\eta, \lambda, \upsilon)$, of the KE distribution are given, respectively, by

$$K(z;\Theta) = \eta\lambda\upsilon e^{-\eta z}\left(1 - e^{-\eta z}\right)^{\lambda-1}\left(1 - \left(1 - e^{-\eta z}\right)^{\lambda}\right)^{\upsilon-1}, \qquad z, \eta, \lambda, \upsilon > 0, \tag{1.3}$$

and

$$k(z;\Theta) = 1 - \left(1 - \left(1 - e^{-\eta z}\right)^{\lambda}\right)^{\upsilon}, \qquad z, \eta, \lambda, \upsilon > 0, \tag{1.4}$$

where, η is the scale parameter, λ, and υ is the scale parameter. Numerous authors have discussed many generlizations of the KE model such as; exponentiated KE, beta KE, truncated bivariate KE, sine KE, Kumaraswamy extended exponential, Topp-Leone KE, gamma KE and Kavya-Manoharan KE models in [41, 42, 43, 44, 45, 46, 47].

In this paper, a novel model of life expectancy is presented by merging the AT-X with the KE distribution. The newly proposed distribution is called the ATKE distribution. These factors provide sufficient justification to explore the proposed paradigm. We define it as described below:

1. The ATKE distribution is highly adaptable, as its PDF can be decreasing, unimodal or tilted to the right. Moreover, the hazard rate function (HRF) shapes of the ATKE model can be decreasing or increasing, increasing-constant, and reversed j-shaped as we have shown in Figures 1 and 2.
2. The newly suggested model incorporates a closed form for the quantile function, which simplifies the calculation of a number of characteristics, such as the generation of random numbers based on the ATKE distribution
3. Some basic features of the suggested ATKE model are studied.
4. To estimate the parameters of the ATKE model, six distinct estimation techniques are utilized. These techniques include Cramer von Mises (CM), Anderson-Darling (AD), least squares (LS), weighted LS (WLS), maximum likelihood (ML), and the right-tail AD (RAD).
5. The recommended model's qualifications are assessed using real data of renewable energy sources. The ATKE model is contrasted with seven well-known statistical distributions that are currently in use: KE, beta Weibull (BWe) [48], length-biased truncated Lomax Weibull (LBTLoWe) [49], new modified Weibull (NMWe) [50], inverse-power logistic-exponential (IPLEx) [51], and half-logistic modified Kies exponential (HLMKE) [52], transmuted MWe (TMWe) [53], modified Weibull (MWe) [54], exponentiated exponential Weibull (EExWe) [55] models. The ATKE model offers a much better fit compablack with other suggested models.

This article is structublack as follows: In Section 2, we discuss the way in which the ATKE distribution is constructed. In Section 3, two significant PDF expansions are calculated. The statistical and mathematical properties of the ATKE distribution are explained in Section 4. Some metrics of uncertainty for the ATKE distribution are discussed in Section 5. The six estimating techniques utilized in Section 6 to estimate the ATKE distribution's parameters. The simulation experiment aimed to assess how accurate the estimates in Section 7 were. The flexibility of the proposed distribution is examined in Section 8 by applying it to two actual data sets. Section 9 concludes with some final remarks.

2. The Arctan Kumaraswamy Exponential Distribution

In this section, we create a new three parameter extension of KE distribution so called the arctan Kumaraswamy exponential (ATKE) distribution. By substituting Equation (1.3) as a baseline CDF in

Equation (1.1), the CDF of the ATKE distribution is as follows:

$$F(z;\Theta) = \frac{4}{\pi}arctan\Big(1-\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon}\Big), z,\eta,\lambda,\upsilon > 0. \tag{2.1}$$

The PDF of the ATKE distribution related to (2.1) is given as follows:

$$f(z;\Theta) = \frac{4\eta\lambda\upsilon e^{-\eta z}(1-e^{-\eta z})^{\lambda-1}\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon-1}}{\pi\Big[1+\Big(1-\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon}\Big)^{2}\Big]}, z,\eta,\lambda,\upsilon > 0, \tag{2.2}$$

The survival function (SF), HRF, and cumulative HRFs for ATKE distribution are provided via

$$S(z;\Theta) = 1-\frac{4}{\pi}arctan\Big(1-\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon}\Big),$$

$$h(z;\Theta) = \frac{4\eta\lambda\upsilon e^{-\eta z}(1-e^{-\eta z})^{\lambda-1}\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon-1}}{\pi\Big[1+\Big(1-\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon}\Big)^{2}\Big]\Big[1-\frac{4}{\pi}arctan\Big(1-\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon}\Big)\Big]},$$

and

$$H(z;\eta,\lambda,\upsilon) = -log\Big[1-\frac{4}{\pi}arctan\Big(1-\Big(1-(1-e^{-\eta z})^{\lambda}\Big)^{\upsilon}\Big)\Big].$$

Figures 1 and 2 show the plots of the PDF and HRF for the ATKE distribution. Figure 1 illustrates the evident that the probability curve is flexible based on the parameter values, suggesting that the distribution will adapt to various sets of data well. Figure 2 shows the hazard curves for various parameter values for the proposed model. It is found that the hazard curve has a variable form based on the parameter values. The curve is inverted bathtub shaped, decreasing, increasing-constant, and decreasing.

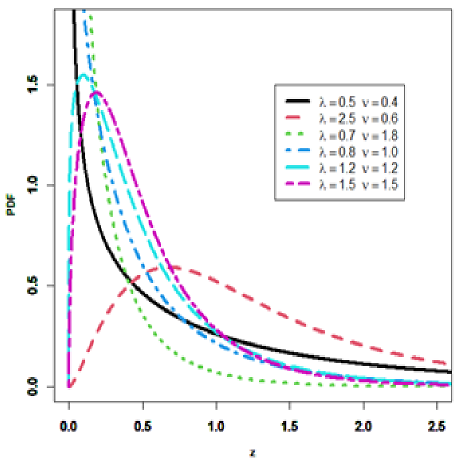

Figure 1. Plots of PDF for the ATKE distribution.

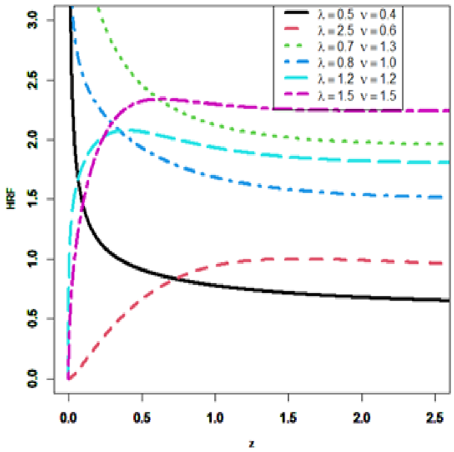

Figure 2. Plots of HRF for the ATKE distribution.

3. Important Expansions

In this section, two significant PDF expansions for the ATKE distribution are provided. Now examine the following binomial expansion $(1+z)^{-1} = \sum_{i=0}^{\infty} (-1)^i z^i$ to PDF (6), we have

$$f(z;\Theta) = \frac{4}{\pi}\eta\lambda\upsilon \sum_{i=0}^{\infty} (-1)^i e^{-\eta z}(1 - e^{-\eta z})^{\lambda-1}\left(1 - (1 - e^{-\eta z})^{\lambda}\right)^{\upsilon-1}\left(1-\left(1 - (1 - e^{-\eta z})^{\lambda}\right)^{\upsilon}\right)^{2i}, \quad (3.1)$$

The binomial expansion is employed to the last term in (7), then it can be reformed as follows:

$$f(z;\Theta) = \frac{4}{\pi}\eta\lambda\upsilon \sum_{i=0}^{\infty}\sum_{j=0}^{2i} (-1)^{i+j}\binom{2i}{j} e^{-\eta z}(1 - e^{-\eta z})^{\lambda-1}\left(1 - (1 - e^{-\eta z})^{\lambda}\right)^{\upsilon(j+1)-1}. \quad (3.2)$$

By applying the binomial series two times in Equation (8), then we can rewrite it as follows

$$f(z;\Theta) = \sum_{i,k,m=0}^{\infty} \Psi_{i,j,k,m} e^{-(m+1)\eta z}, \quad (3.3)$$

where, $\Psi_{i,j,k,m} = \frac{4}{\pi}\sum_{j=0}^{2i} (-1)^{i+j+k+m}\eta\lambda\upsilon \binom{2i}{j}\binom{\upsilon(j+1)-1}{k}\binom{\lambda(k+1)-1}{m}$.

Another important expansion to get the series of $[f(z;\Theta)]^{\gamma}$ by using the binomial expansion

$$[f(z;\Theta)]^{\gamma} = \left(\frac{4\eta\lambda\upsilon}{\pi}\right)^{\gamma} e^{-\gamma\eta z}(1 - e^{-\eta z})^{\gamma\lambda-\gamma}\left(1 - (1 - e^{-\eta z})^{\lambda}\right)^{\upsilon\gamma-\gamma} \times \sum_{i=0}^{\infty}(-1)^i\binom{\gamma+i-1}{i}\left(1-\left(1 - (1 - e^{-\eta z})^{\lambda}\right)^{\upsilon}\right)^{2i}.$$

Then by using the binomial expansion more times we can rewrite the above equation as follows

$$[f(z;\eta,\lambda,\upsilon)]^{\gamma} = \sum_{i,j,k,m=0}^{\infty} \varpi_{i,j,k,m} e^{-(\gamma+m)\eta z}, \quad (3.4)$$

where, $\varpi_{i,j,k,m} = \left(\frac{4\eta\lambda\upsilon}{\pi}\right)^{\gamma}\sum_{j=0}^{2i}(-1)^{i+j+k}\binom{\gamma+i-1}{i}\binom{2i}{j}\binom{\upsilon(\gamma+i)-\gamma}{k}\binom{\lambda(\gamma+k)-\gamma}{m}$.

4. Statistical Features of the ATKE Distribution

In this section, we compute some statistical and mathematical features of the ATKE model.

4.1. Quantile Function and Median

The model's QF, which is an alternative to the distribution function, facilitates a deeper examination of several characteristics, including moments, dispersion, and central tendency. The model's QF is specified by using equation (5).

$$z_q = \frac{-1}{\eta}\log\left[1-\left(1-\left(1-\tan\left(\frac{\pi q}{4}\right)\right)^{\frac{1}{\upsilon}}\right)^{\frac{1}{\lambda}}\right]. \tag{4.1}$$

If $q \in (0,1)$ then $Z \in ATKE$ distribution, the quantile function of ATKE distribution is provided via

$$Q(q) = \frac{-1}{\eta}\log\left[1-\left(1-\left(1-\tan\left(\frac{\pi q}{4}\right)\right)^{\frac{1}{\upsilon}}\right)^{\frac{1}{\lambda}}\right]. \tag{4.2}$$

Also, the median of the ATKE distribution can be computed by inserting q=0.5 in (12) as below

$$Q(q) = \frac{-1}{\eta}\log\left[1-\left(1-\left(1-\tan\left(\frac{\pi}{8}\right)\right)^{\frac{1}{\upsilon}}\right)^{\frac{1}{\lambda}}\right].$$

4.2. Various Measures of Moments

Estimating measures of variation, such as variance (σ^2), standard deviation (σ), coefficient of variation (CV), mean deviation, and median deviation, as well as measurements of shapes, such as coefficient kurtosis (CK) and coefficient skewness (CS), among others, depend on the moments of a distribution. Here several moments measures, including moments, incomplete moments, conditional moments, and moment generating function are determined.

The r^{th} moment of the ATKE model is computed as follows

$$\mu'_r = \int_0^\infty z^r f(z;\Theta)dz = \sum_{i,k,m=0}^{\infty} \Psi_{i,j,k,m} \int_0^\infty z^r e^{-(m+1)\eta z} dz.$$

By using the gamma function to solve the above integral, then

$$\mu'_r = \sum_{i,k,m=0}^{\infty} \frac{\Psi_{i,j,k,m}\Gamma(r+1)}{(m+1)^{r+1}\eta^{r+1}}. \tag{4.3}$$

The first four moments of the ATKE model is easily obtained by inserting r=1, 2, 3 and 4 in (13).

The r^{th} incomplete moments of the ATKE model are computed as follows

$$\Upsilon_r(t) = \int_0^t z^r f(z;\eta,\lambda,\upsilon)dz = \sum_{i,k,m=0}^{\infty} \Psi_{i,j,k,m} \int_0^t z^r e^{-(m+1)\eta z} dz.$$

By using the lower incomplete gamma function to solve the above integral, then

$$\Upsilon_r(t) = \sum_{i,k,m=0}^{\infty} \frac{\Psi_{i,j,k,m}\gamma(r+1,(m+1)\eta t)}{(m+1)^{r+1}\eta^{r+1}}.$$

By the same way, the s^{th} conditional moment of the ATKE model is given by

$$\varsigma_s(t) = \int_t^{\infty} z^s f(z;\eta,\lambda,\upsilon)dz = \sum_{i,j,k,m=0}^{\infty} \Psi_{i,j,k,m} \int_t^{\infty} z^s e^{-(m+1)\eta z} dz$$
$$= \sum_{i,j,k,m=0}^{\infty} \frac{\Psi_{i,j,k,m}\Gamma(s+1,(m+1)\eta t)}{(m+1)^{s+1}\eta^{s+1}}.$$

The moment generating function of Z is provided via

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu'_r = \sum_{r,i,k,m=0}^{\infty} \frac{t^r}{r!}\frac{\Psi_{i,j,k,m}\Gamma(r+1)}{(m+1)^{r+1}\eta^{r+1}}.$$

Some statistical calculations of the first four moments, CS, CK, and CV for the ATKE model for some values of parameters are included in Tables 1 and 2. From these results we infer that the ATKE distribution is right skewed behavior according to skewness values. Also, the ATKE distribution is platykurtic behavior according to kurtosis values.

Tables 1 and 2 shows how statistical values change with varying parameters η, λ, and υ. μ'_1, μ'_2, μ'_3, μ'_4 and σ increase with higher υ and λ. Where CS, CK, and CV decreases with higher υ and λ.

4.3. Order Statistics

The order statistics (OS) is useful in many applications of applied statistics and probability theory. Thus, we have displayed certain OS characteristics for the recommended distribution. Suppose that $Z_1, Z_2, \ldots, Z_n$ be identically independent distributed each with F(z) and $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ be the OS. The PDF of u^{th} OS, that is $Z_{(u)}$, is provided via

$$f_{Z_{(u)}}(z) = \frac{n!}{(u-1)!(n-u)!} F^{u-1}(z) f(z) (1-F(z))^{n-u}. \tag{4.4}$$

The PDF of $Z_{(u)}$ of the ATKE distribution can be determined by inserting (5) and (6) in (14), as follows:

$$f_{Z_{(u)}}(z) = \frac{(n!)4\eta\lambda\upsilon}{(u-1)!(n-u)!}\left(\frac{4}{\pi}\arctan\left(1-(1-\Delta(\eta,\lambda))^{\upsilon}\right)\right)^{u-1}$$
$$\times \frac{e^{-\eta z}\left[\Delta(\eta,\lambda)\right]^{1-\frac{1}{\lambda}}(1-\Delta(\eta,\lambda))^{\upsilon-1}}{\pi\left[1+(1-(1-\Delta(\eta,\lambda))^{\upsilon})^2\right]}\left(1-\frac{4}{\pi}\arctan\left(1-(1-\Delta(\eta,\lambda))^{\upsilon}\right)\right)^{n-u}.$$

where, $\Delta(\eta,\lambda) = (1-e^{-\eta z})^{\lambda}$. Specially, The PDF of the smallest and largest order statistics are provided via

$$f_{Z_{(1)}}(z) = \frac{4n\eta\lambda\upsilon e^{-\eta z}\left[\Delta(\eta,\lambda)\right]^{1-\frac{1}{\lambda}}(1-\Delta(\eta,\lambda))^{\upsilon-1}}{\pi\left[1+(1-(1-\Delta(\eta,\lambda))^{\upsilon})^2\right]}\left(1-\frac{4}{\pi}\arctan\left(1-(1-\Delta(\eta,\lambda))^{\upsilon}\right)\right)^{n-1},$$

and

$$f_{Z_{(n)}}(z) = \frac{4^n n\eta\lambda\upsilon e^{-\eta z}\left[\Delta(\eta,\lambda)\right]^{1-\frac{1}{\lambda}}(1-\Delta(\eta,\lambda))^{\upsilon-1}}{\pi^n\left[1+(1-(1-\Delta(\eta,\lambda))^{\upsilon})^2\right]}\left(\arctan\left(1-(1-\Delta(\eta,\lambda))^{\upsilon}\right)\right)^{n-1}.$$

respectively.

Table 1. Some moments results associated with the ATKE distribution where η=1.1

λ	υ	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CS	CK	CV
0.3	0.1	0.2683	1.1545	9.3089	107.2252	1.0826	7.4738	83.3815	3.8783
	0.3	0.8453	3.8042	30.9644	357.8763	3.0898	4.1476	28.069	2.0796
	0.5	1.3804	6.5518	53.9501	625.8258	4.6462	3.2031	18.1564	1.5615
	0.7	1.8594	9.2973	77.5616	903.3854	5.84	2.732	14.1771	1.2997
	0.9	2.287	12.0008	101.4934	1187.34	6.7706	2.4453	12.0728	1.1378
	1.1	2.6707	14.6438	125.5701	1475.85	7.5109	2.2513	10.7858	1.0262
	1.3	3.0178	17.2184	149.6795	1767.71	8.1113	2.1108	9.924	0.9438
	1.5	3.3339	19.722	173.7461	2062.03	8.6068	2.0042	9.3097	0.88
	1.7	3.6239	22.1547	197.7175	2358.19	9.0219	1.9205	8.8514	0.8288
	1.9	3.8915	24.5182	221.5568	2655.68	9.3742	1.853	8.4972	0.7868
0.5	0.1	0.161	0.4156	2.0107	13.8964	0.3897	7.4738	83.3815	3.8783
	0.3	0.5072	1.3695	6.6883	46.3808	1.1123	4.1476	28.069	2.0796
	0.5	0.8283	2.3586	11.6532	81.107	1.6726	3.2031	18.1564	1.5615
	0.7	1.1156	3.347	16.7533	117.0788	2.1024	2.732	14.1772	1.2997
	0.9	1.3722	4.3203	21.9226	153.8796	2.4374	2.4454	12.0732	1.1378
	1.1	1.6024	5.2718	27.1231	191.2707	2.7039	2.2513	10.7858	1.0262
	1.3	1.8107	6.1986	32.3308	229.0947	2.9201	2.1108	9.924	0.9438
	1.5	2.0004	7.0999	37.5292	267.2395	3.0984	2.0042	9.3097	0.88
	1.7	2.1744	7.9757	42.707	305.6215	3.2479	1.9205	8.8514	0.8288
	1.9	2.3349	8.8266	47.8563	344.1761	3.3747	1.853	8.4972	0.7868
0.8	0.1	0.1006	0.1624	0.4909	2.1204	0.1522	7.4738	83.3817	3.8783
	0.3	0.317	0.535	1.6329	7.0771	0.4345	4.1476	28.069	2.0796
	0.5	0.5177	0.9213	2.845	12.376	0.6534	3.2031	18.1564	1.5615
	0.7	0.6973	1.3074	4.0902	17.8648	0.8213	2.732	14.1771	1.2997
	0.9	0.8576	1.6876	5.3522	23.4802	0.9521	2.4453	12.0728	1.1378
	1.1	1.0015	2.0593	6.6219	29.1856	1.0562	2.2513	10.7858	1.0262
	1.3	1.1317	2.4213	7.8933	34.9571	1.1407	2.1108	9.924	0.9438
	1.5	1.2502	2.7734	9.1624	40.7775	1.2103	2.0042	9.3097	0.88
	1.7	1.359	3.1155	10.4265	46.6341	1.2687	1.9205	8.8514	0.8288
	1.9	1.4593	3.4479	11.6837	52.5171	1.3183	1.853	8.4972	0.7868
1.1	0.1	0.0732	0.0859	0.1888	0.5932	0.0805	7.4738	83.3815	3.8783
	0.3	0.2305	0.283	0.6281	1.9799	0.2298	4.1476	28.0691	2.0796
	0.5	0.3765	0.4873	1.0944	3.4623	0.3456	3.2031	18.1564	1.5615
	0.7	0.5071	0.6915	1.5734	4.9979	0.4344	2.732	14.1771	1.2997
	0.9	0.6237	0.8926	2.0588	6.5689	0.5036	2.4453	12.0728	1.1378
	1.1	0.7284	1.0892	2.5473	8.1651	0.5587	2.2513	10.7859	1.0262
	1.3	0.823	1.2807	3.0363	9.7797	0.6033	2.1108	9.924	0.9438
	1.5	0.9093	1.4669	3.5245	11.408	0.6402	2.0042	9.3097	0.88
	1.7	0.9883	1.6479	4.0108	13.0465	0.6711	1.9205	8.8514	0.8288
	1.9	1.0613	1.8237	4.4944	14.6923	0.6973	1.853	8.4972	0.7868

Table 2. Some moments results associated with the ATKE distribution where η=1.5

λ	υ	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CS	CK	CV
0.3	0.1	0.1076	0.2941	1.6191	13.1357	0.2825	10.1679	156.1189	4.9392
	0.3	0.4758	1.4286	8.1217	66.9153	1.2022	4.778	36.8413	2.3046
	0.5	0.8839	2.8977	16.9477	141.4834	2.1165	3.4573	20.8325	1.646
	0.7	1.279	4.5412	27.2894	230.616	2.9054	2.8369	15.1103	1.3327
	0.9	1.6478	6.2795	38.7274	331.1202	3.5643	2.4718	12.2828	1.1458
	1.1	1.9883	8.066	50.9918	440.931	4.1128	2.2299	10.631	1.02
	1.3	2.3022	9.8722	63.8921	558.5638	4.572	2.0574	9.5612	0.9288
	1.5	2.5923	11.6798	77.2876	682.8864	4.96	1.9279	8.8181	0.8591
	1.7	2.8612	13.4772	91.0708	812.9981	5.2908	1.827	8.275	0.8039
	1.9	3.1115	15.2568	105.1581	948.1664	5.5758	1.7462	7.8626	0.7589
0.5	0.1	0.0646	0.1059	0.3497	1.7024	0.1017	10.1679	156.1189	4.9391
	0.3	0.2855	0.5143	1.7543	8.6722	0.4328	4.778	36.8414	2.3046
	0.5	0.5303	1.0432	3.6607	18.3362	0.7619	3.4573	20.8325	1.646
	0.7	0.7674	1.6349	5.8945	29.8878	1.0459	2.8369	15.1103	1.3327
	0.9	0.9887	2.2606	8.3651	42.9132	1.2832	2.4718	12.2828	1.1458
	1.1	1.193	2.9038	11.0142	57.1447	1.4806	2.2299	10.631	1.02
	1.3	1.3813	3.554	13.8007	72.3899	1.6459	2.0574	9.5612	0.9288
	1.5	1.5554	4.2047	16.6941	88.5021	1.7856	1.9279	8.8181	0.8591
	1.7	1.7167	4.8518	19.6713	105.3646	1.9047	1.827	8.275	0.8039
	1.9	1.8669	5.4925	22.7142	122.8824	2.0073	1.7462	7.8626	0.7589
0.8	0.1	0.0404	0.0414	0.0854	0.2598	0.0397	10.1678	156.1179	4.9393
	0.3	0.1784	0.2009	0.4283	1.3233	0.1691	4.778	36.8415	2.3046
	0.5	0.3314	0.4075	0.8937	2.7979	0.2976	3.4573	20.8325	1.646
	0.7	0.4796	0.6386	1.4391	4.5605	0.4086	2.8369	15.1103	1.3327
	0.9	0.6179	0.8831	2.0423	6.548	0.5012	2.4718	12.2828	1.1458
	1.1	0.7456	1.1343	2.689	8.7196	0.5784	2.2299	10.631	1.02
	1.3	0.8633	1.3883	3.3693	11.0458	0.6429	2.0574	9.5612	0.9288
	1.5	0.9721	1.6425	4.0757	13.5043	0.6975	1.9279	8.8181	0.8591
	1.7	1.0729	1.8952	4.8026	16.0774	0.744	1.827	8.275	0.8039
	1.9	1.1668	2.1455	5.5455	18.7504	0.7841	1.7462	7.8626	0.7589
1.1	0.1	0.0294	0.0219	0.0328	0.0727	0.021	10.1678	156.1182	4.9394
	0.3	0.1298	0.1063	0.1648	0.3702	0.0894	4.7782	36.8428	2.3045
	0.5	0.2411	0.2155	0.3438	0.7827	0.1574	3.4573	20.8324	1.646
	0.7	0.3488	0.3378	0.5536	1.2759	0.2161	2.8369	15.1101	1.3327
	0.9	0.4494	0.4671	0.7856	1.8319	0.2651	2.4718	12.2828	1.1458
	1.1	0.5423	0.6	1.0344	2.4394	0.3059	2.2299	10.631	1.02
	1.3	0.6279	0.7343	1.2961	3.0902	0.3401	2.0574	9.5612	0.9288
	1.5	0.707	0.8687	1.5678	3.7781	0.3689	1.9279	8.8187	0.8591
	1.7	0.7803	1.0024	1.8474	4.4978	0.3935	1.827	8.275	0.8039
	1.9	0.8486	1.1348	2.1332	5.2456	0.4147	1.7462	7.8626	0.7589

5. Different Measures of Uncertainty

One of the most known measures of uncertainty is entropy. The entropy of the ATKE distribution can be measublack by various measures as Rényi entropy (RE) [55], q entropy (QE) [56] Arimoto entropy (AE) [57] and Havrda and Charvat entropy (HCE) by [58]. The entropy of a random variable Z is a measure of an uncertainty and has PDF $f(z)$. The four metrics of uncertainty are shown in Table 3.

Table 3. Different Measures of Uncertainty

measures	Formula
RE	$I_R(\gamma) = \frac{1}{1-\gamma} log\left[\int_0^\infty f^\gamma(z;\Theta)\,dz\right], \quad \gamma \neq 1, \quad \gamma > 0.$
QE	$Q_R(\gamma) = \frac{1}{\gamma-1}\left[1 - \int_0^\infty f^\gamma(z;\Theta)\,dz\right], \quad \gamma \neq 1, \quad \gamma > 0.$
AE	$A_R(\gamma) = \frac{\gamma}{1-\gamma}\left[\left(\int_0^\infty f^\gamma(z;\Theta)\,dz\right)^{\frac{1}{\gamma}} - 1\right], \quad \gamma \neq 1, \quad \gamma > 0.$
HCE	$HC_R(\gamma) = \frac{1}{2^{1-\gamma}-1}\left[\int_0^\infty f^\gamma(z;\Theta)\,dz - 1\right], \quad \gamma \neq 1, \quad \gamma > 0.$

We now want to compute the subsequent integral $I = \int_0^\infty [f(z;\Theta)]^\gamma\,dz$ as below

$$I = \int_0^\infty [f(z;\Theta)]^\gamma\,dz = \sum_{i,j,k,m=0}^{\infty} \varpi_{i,j,k,m} \int_0^\infty e^{-(\gamma+m)\eta z} dz = \sum_{i,j,k,m=0}^{\infty} \frac{\varpi_{i,j,k,m}}{(\gamma+m)\eta}.$$

By inserting the value of the preceding integral I in the entropy measures provided in Table 3, the formulae of these measures are obtained for the ATKE distribution.

Some numerical values for REM QE, AE, and HACE of the ATKE distribution for $\gamma = 0.1$ or 0.5 are listed in Tables 4 and 5.

Both tables 4 and 5 exhibit a consistent trend where increasing λ or ν leads to higher (more negative) entropy values. Comparison between η values Comparing the tables, the absolute values of entropy measures for $\eta = 1.5$ are generally higher (more negative) than those for $\eta = 1.1$.

6. Different Estimation Methods

In this section, we will find different estimators, namely ML estimator (MLE), LS estimator (LSE), WLS estimator (WLSE), CM estimator (CME), AD estimator (ADE) and the RAD estimator (RADE) of set of parameters $\Theta = (\eta, \lambda, \upsilon)^T$ of our distribution.

6.1. Maximum Likelihood Estimation Method

The ML method is most frequently used in estimating the unknown parameters, it depends on maximizing the logarithm of the likelihood of the distribution. Let $z_1, ..., z_n$ is a random sample has size n of the ATKED distribution the log-likelihood function is,

Table 4. Numerical values of entropy measures of the ATKE distribution where η=1.1

λ	υ	γ =0.1				γ =0.5			
		RE	QE	AE	HACE	RE	QE	AE	HACE
0.3	0.1	-0.0788	-0.1674	-0.0894	-0.1739	-0.6811	-1.087	-0.7916	-1.3121
	0.3	-0.0439	-0.0965	-0.0663	-0.1003	-0.3506	-0.6643	-0.554	-0.8019
	0.5	-0.0378	-0.0837	-0.0603	-0.087	-0.2758	-0.5441	-0.4701	-0.6568
	0.7	-0.0394	-0.0871	-0.062	-0.0905	-0.2678	-0.5307	-0.4603	-0.6406
	0.9	-0.0441	-0.0971	-0.0666	-0.1009	-0.2872	-0.5631	-0.4838	-0.6797
	1.1	-0.0505	-0.1105	-0.0721	-0.1148	-0.3201	-0.6166	-0.5215	-0.7442
	1.3	-0.0579	-0.1256	-0.0776	-0.1305	-0.3605	-0.6794	-0.564	-0.8202
	1.5	-0.0659	-0.1417	-0.0827	-0.1473	-0.4053	-0.7458	-0.6068	-0.9003
	1.7	-0.0743	-0.1585	-0.0873	-0.1647	-0.4527	-0.8124	-0.6474	-0.9807
	1.9	-0.0831	-0.1757	-0.0912	-0.1826	-0.5017	-0.8775	-0.685	-1.0592
0.5	0.1	-0.083	-0.1755	-0.0912	-0.1824	-0.7323	-1.1392	-0.8148	-1.3752
	0.3	-0.0424	-0.0935	-0.065	-0.0972	-0.3583	-0.676	-0.5617	-0.816
	0.5	-0.031	-0.0692	-0.0527	-0.0719	-0.2428	-0.4877	-0.4282	-0.5887
	0.7	-0.0279	-0.0623	-0.0487	-0.0648	-0.1991	-0.4096	-0.3677	-0.4944
	0.9	-0.0283	-0.0634	-0.0493	-0.0659	-0.1866	-0.3866	-0.3493	-0.4667
	1.1	-0.0308	-0.0686	-0.0524	-0.0713	-0.1904	-0.3937	-0.355	-0.4753
	1.3	-0.0343	-0.0763	-0.0566	-0.0793	-0.2037	-0.418	-0.3743	-0.5046
	1.5	-0.0387	-0.0855	-0.0612	-0.0889	-0.2228	-0.4524	-0.4013	-0.5461
	1.7	-0.0435	-0.0958	-0.066	-0.0996	-0.2456	-0.4926	-0.432	-0.5947
	1.9	-0.0488	-0.1068	-0.0707	-0.1109	-0.271	-0.536	-0.4642	-0.6471
0.8	0.1	-0.0914	-0.1917	-0.0944	-0.1992	-0.8196	-1.2216	-0.8485	-1.4746
	0.3	-0.0465	-0.1021	-0.0687	-0.1061	-0.4126	-0.7562	-0.6133	-0.9129
	0.5	-0.0307	-0.0685	-0.0523	-0.0712	-0.2647	-0.5253	-0.4563	-0.6341
	0.7	-0.0234	-0.0526	-0.0427	-0.0547	-0.1917	-0.3961	-0.3569	-0.4781
	0.9	-0.0201	-0.0454	-0.0379	-0.0472	-0.1531	-0.3232	-0.2971	-0.3901
	1.1	-0.019	-0.043	-0.0362	-0.0447	-0.1333	-0.2845	-0.2643	-0.3434
	1.3	-0.0193	-0.0436	-0.0367	-0.0454	-0.1248	-0.2677	-0.2498	-0.3232
	1.5	-0.0205	-0.0463	-0.0385	-0.0481	-0.1237	-0.2655	-0.2479	-0.3205
	1.7	-0.0224	-0.0503	-0.0412	-0.0523	-0.1276	-0.2732	-0.2546	-0.3298
	1.9	-0.0247	-0.0554	-0.0445	-0.0576	-0.135	-0.2878	-0.2671	-0.3474
1.1	0.1	-0.101	-0.2098	-0.0974	-0.218	-0.9087	-1.2974	-0.8766	-1.5661
	0.3	-0.0538	-0.1171	-0.0746	-0.1217	-0.4833	-0.8534	-0.6714	-1.0302
	0.5	-0.0354	-0.0786	-0.0578	-0.0817	-0.3166	-0.6109	-0.5176	-0.7374
	0.7	-0.0256	-0.0575	-0.0458	-0.0598	-0.2261	-0.4584	-0.4058	-0.5533
	0.9	-0.02	-0.0452	-0.0377	-0.0469	-0.1716	-0.3585	-0.3264	-0.4327
	1.1	-0.0168	-0.0379	-0.0326	-0.0394	-0.1373	-0.2924	-0.2711	-0.353
	1.3	-0.015	-0.034	-0.0297	-0.0354	-0.1157	-0.2493	-0.2338	-0.301
	1.5	-0.0143	-0.0324	-0.0284	-0.0336	-0.1024	-0.2224	-0.21	-0.2684
	1.7	-0.0142	-0.0323	-0.0284	-0.0336	-0.0949	-0.2071	-0.1964	-0.25
	1.9	-0.0147	-0.0334	-0.0292	-0.0347	-0.0917	-0.2004	-0.1903	-0.2419

Table 5. Numerical values of entropy measures of the ATKE distribution where η=1.5

λ	υ	γ =0.1				γ =0.5			
		RE	QE	AE	HACE	RE	QE	AE	HACE
0.3	0.1	-0.0701	-0.1503	-0.0851	-0.1562	-0.6053	-1.0038	-0.7519	-1.2117
	0.3	-0.0388	-0.0859	-0.0614	-0.0892	-0.3037	-0.5901	-0.5031	-0.7123
	0.5	-0.0358	-0.0795	-0.0582	-0.0826	-0.2522	-0.504	-0.4405	-0.6084
	0.7	-0.04	-0.0884	-0.0626	-0.0919	-0.2627	-0.5219	-0.4538	-0.63
	0.9	-0.0471	-0.1033	-0.0692	-0.1073	-0.2975	-0.58	-0.4959	-0.7001
	1.1	-0.0556	-0.121	-0.076	-0.1257	-0.3439	-0.6538	-0.5469	-0.7892
	1.3	-0.0651	-0.1402	-0.0823	-0.1457	-0.3963	-0.7327	-0.5985	-0.8845
	1.5	-0.0752	-0.1602	-0.0877	-0.1665	-0.4521	-0.8115	-0.6469	-0.9796
	1.7	-0.0857	-0.1807	-0.0923	-0.1878	-0.5094	-0.8875	-0.6906	-1.0713
	1.9	-0.0965	-0.2014	-0.0961	-0.2093	-0.5675	-0.9594	-0.7293	-1.1581
0.5	0.1	-0.0729	-0.1557	-0.0866	-0.1618	-0.6423	-1.0453	-0.7721	-1.2618
	0.3	-0.0357	-0.0792	-0.0581	-0.0823	-0.2954	-0.5767	-0.4935	-0.6961
	0.5	-0.0274	-0.0613	-0.0481	-0.0637	-0.2036	-0.418	-0.3743	-0.5045
	0.7	-0.0269	-0.0602	-0.0475	-0.0626	-0.1793	-0.3729	-0.3382	-0.4502
	0.9	-0.0297	-0.0664	-0.0511	-0.069	-0.1829	-0.3798	-0.3437	-0.4584
	1.1	-0.0343	-0.0763	-0.0566	-0.0793	-0.2006	-0.4125	-0.37	-0.4979
	1.3	-0.04	-0.0883	-0.0626	-0.0918	-0.2262	-0.4585	-0.4059	-0.5534
	1.5	-0.0463	-0.1016	-0.0685	-0.1056	-0.2563	-0.5111	-0.4458	-0.6169
	1.7	-0.0531	-0.1157	-0.0741	-0.1202	-0.2892	-0.5664	-0.4862	-0.6837
	1.9	-0.0602	-0.1304	-0.0792	-0.1355	-0.3238	-0.6223	-0.5255	-0.7512
0.8	0.1	-0.079	-0.1679	-0.0895	-0.1745	-0.7086	-1.1155	-0.8044	-1.3465
	0.3	-0.0371	-0.0823	-0.0596	-0.0855	-0.3261	-0.6261	-0.5281	-0.7557
	0.5	-0.0242	-0.0544	-0.0438	-0.0565	-0.2013	-0.4138	-0.371	-0.4994
	0.7	-0.0196	-0.0442	-0.0371	-0.0459	-0.1482	-0.3137	-0.2891	-0.3787
	0.9	-0.0187	-0.0422	-0.0357	-0.0439	-0.1265	-0.271	-0.2527	-0.3271
	1.1	-0.0198	-0.0446	-0.0374	-0.0464	-0.1212	-0.2606	-0.2436	-0.3145
	1.3	-0.0221	-0.0497	-0.0408	-0.0517	-0.1256	-0.2693	-0.2512	-0.3251
	1.5	-0.0252	-0.0566	-0.0452	-0.0588	-0.136	-0.2898	-0.2688	-0.3498
	1.7	-0.0289	-0.0646	-0.0501	-0.0671	-0.1502	-0.3177	-0.2925	-0.3835
	1.9	-0.033	-0.0734	-0.055	-0.0763	-0.1671	-0.35	-0.3194	-0.4225
1.1	0.1	-0.0863	-0.182	-0.0926	-0.1892	-0.7769	-1.1823	-0.8329	-1.4272
	0.3	-0.0417	-0.0919	-0.0643	-0.0955	-0.3738	-0.6994	-0.5771	-0.8442
	0.5	-0.026	-0.0582	-0.0463	-0.0605	-0.2291	-0.4636	-0.4099	-0.5596
	0.7	-0.0188	-0.0424	-0.0358	-0.044	-0.1583	-0.3332	-0.3055	-0.4022
	0.9	-0.0155	-0.035	-0.0305	-0.0364	-0.121	-0.2601	-0.2432	-0.314
	1.1	-0.0143	-0.0325	-0.0286	-0.0338	-0.1019	-0.2213	-0.2091	-0.2672
	1.3	-0.0146	-0.033	-0.029	-0.0343	-0.0936	-0.2043	-0.1938	-0.2466
	1.5	-0.0157	-0.0355	-0.0308	-0.0369	-0.0923	-0.2015	-0.1914	-0.2433
	1.7	-0.0174	-0.0394	-0.0337	-0.041	-0.0956	-0.2085	-0.1976	-0.2517
	1.9	-0.0196	-0.0443	-0.0372	-0.0461	-0.1023	-0.2222	-0.2098	-0.2682

$$\begin{aligned} l_n &= \sum_{i=1}^{n} \log(f(z_{(i)};\Theta)) \\ &= n\log(4) + n\log(\eta) + n\log(\lambda) + n\log(\upsilon) - \sum_{i=1}^{n} \eta z_i \\ &+ (\lambda-1)\sum_{i=1}^{n}\log(1-e^{-\eta z_i}) + (\upsilon-1)\sum_{i=1}^{n}\log\left(1-(1-e^{-\eta z_i})^{\lambda}\right) - n\log(\Pi) \\ &- \sum_{i=1}^{n}\log\left(\left[1+\left(1-\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon}\right)^{2}\right]\right) \end{aligned}$$

To maximize l_n first we get the partial derivative with respect to η,λ, and υ

$$\begin{aligned} \frac{\partial L_n}{\partial \upsilon} &= \frac{n}{\upsilon} + \sum_{i=1}^{n}\log(1-e^{-\eta z_i})^{\lambda} \\ &+ \sum_{i=1}^{n} \frac{2\left(1-\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon}\right)\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon}\log\left(1-(1-e^{-\eta z_i})^{\lambda}\right)}{1+(1-(1-(1-e^{-\eta z_i})^{\lambda})^{\upsilon})^{2}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial L_n}{\partial \eta} &= \frac{n}{\eta} - \sum_{i=1}^{n} z_i + (\lambda-1)\sum_{i=1}^{n}\frac{z_i e^{\eta z_i}}{1-e^{\eta z_i}} - (\upsilon-1)\lambda\sum_{i=1}^{n}\frac{z_i e^{\eta z_i}(1-e^{-\eta z_i})^{\lambda-1}}{(1-(1-e^{-\eta z_i})^{\lambda})} \\ &- 2\upsilon\lambda\sum_{i=1}^{n}\frac{z_i e^{\eta z_i}\left(1-\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon}\right)\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon-1}(1-e^{-\eta z_i})^{\lambda-1}}{1+(1-(1-(1-e^{-\eta z_i})^{\lambda})^{\upsilon})^{2}}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L_n}{\partial \lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n}\log(1-e^{-\eta z_i}) - (\upsilon-1)\sum_{i=1}^{n}\frac{(1-e^{-\eta z_i})^{\lambda}\log(1-e^{-\eta z_i})}{1-(1-e^{-\eta z_i})} \\ &+ 2\upsilon\sum_{i=1}^{n}\frac{\left(1-\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon}\right)\left(1-(1-e^{-\eta z_i})^{\lambda}\right)^{\upsilon-1}(1-e^{-\eta z_i})^{\lambda}\log(1-e^{-\eta z_i})}{1+(1-(1-(1-e^{-\eta z_i})^{\lambda})^{\upsilon})^{2}}, \end{aligned}$$

Solving above first order derivatives setting to zero, parameters of the proposed model can be estimated. Solution of above equation is not possible so computer programming can be used.

6.2. Least Squares Method

Suppose that the OS of a random sample from the ATKE distribution, denoted by $z_{(1)}, z_{(2)}, ..., z_{(n)}$. The LSEs of the parameters η,λ, and υ from the ATKE distribution are obtained by minimizing the following function

$$LS(\Theta) = \sum_{i=1}^{n}\left[F(z_{(i)};\Theta) - \frac{1}{n+1}\right]^{2}.$$

Substituting the CDF (1.3) of the ATKE distribution into the above equation yields

$$LS(\Theta) = \sum_{i=1}^{n}\left[\frac{4}{\pi}arctan\left(1-\left(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\right)^{\upsilon}\right) - \frac{1}{n+1}\right]^2.$$

Moreover, the following system of non-linear equations may be solved to get the LSEs of the ATKE distribution parameters

$$\sum_{i=1}^{n}\left[\frac{4}{\pi}arctan\left(1-\left(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\right)^{\upsilon}\right) - \frac{1}{n+1}\right]^2 \varepsilon_1\left(z_{(i)}\,|\eta,\lambda,\upsilon\right) = 0, \tag{6.1}$$

$$\sum_{i=1}^{n}\left[\frac{4}{\pi}arctan\left(1-\left(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\right)^{\upsilon}\right) - \frac{1}{n+1}\right]^2 \varepsilon_2\left(z_{(i)}\,|\eta,\lambda,\upsilon\right) = 0, \tag{6.2}$$

$$\sum_{i=1}^{n}\left[\frac{4}{\pi}arctan\left(1-\left(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\right)^{\upsilon}\right) - \frac{1}{n+1}\right]^2 \varepsilon_3\left(z_{(i)}\,|\eta,\lambda,\upsilon\right) = 0, \tag{6.3}$$

where $\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right)$ are stated in Equations (6.1)-(6.2) for $k = 1, 2, 3$.

6.3. Weighted Least Squares Method

The WLSEs of the parameters η,λ, and υ from the ATKE distribution are obtained by minimizing the following function

$$WS(\Theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(z_{(i)};\Theta) - \frac{1}{n+1}\right]^2.$$

Substituting the CDF (1.3) of the ATKE distribution into the above equation yields

$$WS(\Theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{4}{\pi}arctan\left(1-\left(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\right)^{\upsilon}\right) - \frac{1}{n+1}\right]^2.$$

In addition, the following system of non-linear equations may be solved to yield the WLSEs of the ATKE distribution parameters (for k=1,2,3)

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{4}{\pi}arctan\left(1-\left(1 - (1 - e^{-\eta z_{i}})^{\lambda}\right)^{\upsilon}\right) - \frac{1}{n+1}\right]^2 \varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right) = 0,$$

where $\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right)$ are stated in Equations (6.1)-(6.2) for $k = 1, 2, 3$.

6.4. Cramer von Mises Estimation Method

The CVMEs are classified as minimal distance estimators and they have less bias relative to other estimators of the same kind. Finding the difference between the estimated and empirical CDFs allows one to generate these estimators. The following equation may be minimized with regard to parameters η,λ, and υ in order to get the CVMEs of the ATKE distribution

$$CM(\Theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(z_{(i)};\Theta) - \frac{2i-1}{n+1}\right]^2.$$

Substituting the CDF (1.3) of the ATKE distribution into the above equation yields

$$CM(\Theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{4}{\pi}arctan\Big(1-\Big(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\Big)^{\upsilon}\Big) - \frac{2i-1}{n+1}\right]^2.$$

Similar to this, the CVMEs may also be found by working through the non-linear equation system below (for $k = 1, 2, 3$)

$$\sum_{i=1}^{n}\left[\frac{4}{\pi}arctan\Big(1-\Big(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\Big)^{\upsilon}\Big) - \frac{2i-1}{n+1}\right]^2 \varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right) = 0,$$

where $\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right)$ are stated in Equations (6.1)-(6.2) for $k = 1, 2, 3$.

6.5. Anderson-Darling Method

The AD approach may be used to estimate the parameters of any distribution based on observed data. It quantifies, in particular, the difference between the observed and pblackicted distributions. It considers both how close the actual points are to the theoretical distribution and how significant the distribution's tails are. As minimal distance estimators, the ADEs are included in this group. The ADEs of the ATKE distribution parameters may be obtained by minimizing the following equation in relation to η,λ, and υ.

$$AD(\Theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[logF(z_{(i)};\Theta) + log(1 - F(z_{(n+i-1)};\Theta))\right]^2.$$

Inserting the CDF (1.3) of the ATKE distribution into the above equation gives

$$\begin{aligned} AD(\Theta) &= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\Big(1-\frac{4}{\pi}arctan\Big(1-\Big(1 - (1 - e^{-\eta z_{(i)}})^{\lambda}\Big)^{\upsilon}\Big)\Big)\right. \\ &- \left. log\Big(\frac{4}{\pi}arctan\Big(1-\Big(1 - (1 - e^{-\eta z_{(n-i+1)}}e^{-\eta z_{(i)}})^{\lambda}\Big)^{\upsilon}\Big)\Big)\right]^2. \end{aligned}$$

In accordance with this, the following system of non-linear equations must also be solved in order to get the ADEs for the ATKE distribution parameters (for $k = 1, 2, 3$)

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right)}{F(z_{(i)};\Theta)} - \frac{\varepsilon_k\left(z_{(n-i+1)}\,|\eta,\lambda,\upsilon\right)}{S\left(z_{(n-i+1)};\Theta\right)}\right] = 0,$$

where $\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right)$ are stated in Equations (6.1)-(6.2) for $k = 1, 2, 3$.

6.6. The Right - tail Anderson-Darling Estimation Method

The RADEs of the ATKE distribution parameters may be obtained by minimizing the following equation in relation to η,λ, and υ

$$RA(\Theta) = \frac{n}{2} - 2\sum_{i=1}^{n} F(z_{(i)};\Theta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)log(S(z_{(n+i-1)};\Theta)).$$

In accordance with this, the following system of non-linear equations must also be solved in order to get the ADEs for the ATKE distribution parameters (for $k = 1, 2, 3$)

$$-2\sum_{i=1}^{n}\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right) - \frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)\varepsilon_k\left(z_{(n-i+1)}\,|\eta,\lambda,\upsilon\right)}{S\left(z_{(n+i-1)};\Theta\right)}.$$

where $\varepsilon_k\left(z_{(i)}\,|\eta,\lambda,\upsilon\right)$ are stated in Equations (6.1)-(6.2) for $k = 1, 2, 3$.

7. Simulation

In this section we perform the numerical study using R software to find the optimum method of estimation using the previous methods, to estimate (υ,η,λ) of the ATKE distribution, we generate random samples of different sizes of $n = (25, 50, 75, 100, 125, 150, 175, 200, 225, 250)$ from ATKED with replication 1000 times with different initial values of (υ,η,λ).

The estimates of the mean and mean square error (MSE) exist in Tables 6-9. These tables show:

- The estimated parameters (υ,η,λ) goes closer to their initial values as n increases.
- The values of MSE decreases as n increases.
- To compare between the different methods of estimation use the ranks for all initial values which are summarized in Tables 10.

Table 6. Simulation result for υ=0.3, η=0.3 and λ=0.7

n		υ						η						λ					
		MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE
25	mean	0.846	0.87	0.984	0.7999	0.9603	1.0011	0.495	0.4981	0.5074	0.4863	0.4968	0.3884	0.8639	0.7382	0.7359	0.8905	0.7776	0.8985
	MSE	2.1101	1.5676	2.1117	1.2574	2.1262	1.9137	0.2928	0.3088	0.3373	0.2848	0.2946	0.1972	0.3655	0.2721	0.2127	0.4779	0.1951	0.4955
	Rank	4	2	5	1	6	3	3	5	6	2	4	1	4	3	2	5	1	6
50	mean	0.9458	0.6978	0.8407	0.7271	0.8203	0.8329	0.3906	0.4388	0.4213	0.4294	0.4323	0.3746	0.7733	0.7013	0.7113	0.7608	0.7252	0.7738
	MSE	2.3223	0.7739	1.3261	0.887	1.4159	1.2211	0.1502	0.1928	0.1801	0.19	0.1749	0.1315	0.0711	0.0804	0.0602	0.0976	0.0581	0.0949
	Rank	6	1	4	2	5	3	2	6	4	5	3	1	3	4	2	6	1	5
75	mean	0.9167	0.6884	0.8086	0.6434	0.7723	0.7878	0.3584	0.3991	0.3836	0.4192	0.3958	0.345	0.7385	0.6869	0.7005	0.7374	0.7184	0.759
	MSE	2.0693	0.6851	1.2361	0.5869	1.0757	0.9738	0.1126	0.1442	0.1266	0.1592	0.135	0.1019	0.0339	0.0407	0.0338	0.0605	0.0314	0.0549
	Rank	6	2	5	1	4	3	2	5	3	6	4	1	3	4	2	6	1	5
100	mean	0.9465	0.6222	0.7151	0.5927	0.7007	0.6959	0.3372	0.3972	0.3711	0.4229	0.3752	0.3471	0.7298	0.6921	0.7049	0.7095	0.7087	0.736
	MSE	2.1127	0.484	0.8124	0.4785	0.826	0.7491	0.0911	0.1343	0.1067	0.1502	0.1067	0.0878	0.02	0.0312	0.0235	0.034	0.0225	0.0367
	Rank	6	2	4	1	5	3	2	5	3	6	3	1	1	4	3	5	2	6
125	mean	0.8749	0.5824	0.6619	0.5804	0.6675	0.6455	0.3408	0.4063	0.3743	0.4012	0.3591	0.3278	0.725	0.6859	0.7003	0.7057	0.7063	0.7269
	MSE	1.827	0.427	0.6714	0.4205	0.6411	0.516	0.0848	0.1339	0.1019	0.1322	0.0922	0.0749	0.0197	0.028	0.0217	0.0258	0.0177	0.0282
	Rank	6	2	5	1	4	3	2	6	4	5	3	1	2	5	3	4	1	6
150	mean	0.9291	0.5538	0.6486	0.5463	0.6483	0.6421	0.3211	0.3863	0.3534	0.4195	0.3553	0.3357	0.7232	0.6898	0.7034	0.7034	0.71	0.7271
	MSE	1.9711	0.3212	0.5739	0.3617	0.6113	0.5141	0.0714	0.1159	0.0867	0.1369	0.0839	0.0755	0.0142	0.0221	0.0165	0.0221	0.0155	0.0214
	Rank	6	1	4	2	5	3	1	5	4	6	3	2	1	5	3	6	2	4
175	mean	0.8577	0.561	0.6116	0.5048	0.5902	0.5633	0.3159	0.377	0.3538	0.4043	0.3492	0.3421	0.7201	0.6882	0.6981	0.6988	0.7049	0.7187
	MSE	1.6703	0.3422	0.4996	0.2665	0.4419	0.3488	0.0631	0.1097	0.0801	0.1202	0.0744	0.0673	0.0126	0.02	0.0148	0.02	0.0135	0.0193
	Rank	6	2	5	1	4	3	1	5	4	6	3	2	1	5	3	6	2	4
200	mean	0.7885	0.5155	0.5724	0.4997	0.5743	0.5722	0.312	0.3892	0.3476	0.4059	0.3429	0.3306	0.7215	0.689	0.7042	0.6966	0.7059	0.718
	MSE	1.3271	0.2557	0.3757	0.2422	0.4196	0.3824	0.0574	0.1116	0.0722	0.121	0.0678	0.0631	0.0107	0.017	0.0121	0.018	0.0124	0.0173
	Rank	6	2	3	1	5	4	1	5	4	6	3	2	1	4	2	6	3	5
225	mean	0.8459	0.521	0.5938	0.4662	0.5352	0.529	0.3105	0.3799	0.3383	0.4145	0.3474	0.3398	0.7205	0.6929	0.707	0.6909	0.7038	0.7136
	MSE	1.5629	0.2484	0.4023	0.1977	0.3067	0.2702	0.0575	0.1057	0.0718	0.1176	0.0638	0.0598	0.0106	0.0156	0.0116	0.0166	0.0111	0.0151
	Rank	6	2	5	1	4	3	1	5	4	6	3	2	1	5	3	6	2	4
250	mean	0.7687	0.4717	0.5331	0.4813	0.544	0.5186	0.3158	0.4106	0.3535	0.3959	0.3427	0.3423	0.7183	0.6807	0.7007	0.6969	0.7072	0.7138
	MSE	1.345	0.2144	0.3244	0.202	0.3333	0.2641	0.0505	0.1122	0.0676	0.1079	0.0621	0.0596	0.01	0.0161	0.0115	0.0142	0.0102	0.0131
	Rank	6	2	4	1	5	3	1	6	4	5	3	2	1	6	3	5	2	4
Sum of ranks		58	18	44	12	47	31	16	53	40	53	32	15	18	45	26	55	17	49

Table 7. Simulation result for υ=0.3, η=0.3 and λ=1.2

n		υ						η						λ					
		MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE
25	mean	0.7288	0.7613	0.8454	0.8527	0.9962	1.0488	0.5225	0.417	0.4397	0.4315	0.4087	0.3406	1.8979	1.476	1.4667	1.7721	1.5011	1.6536
	MSE	1.8018	1.05	1.3955	1.3389	2.0484	1.9519	0.2766	0.1759	0.2017	0.2048	0.1874	0.1281	3.6506	2.6996	2.2011	3.5061	1.7396	3.5428
	Rank	4	1	3	2	6	5	6	2	4	5	3	1	6	3	2	4	1	5
50	mean	0.9424	0.6655	0.8038	0.7056	0.8051	0.8192	0.3913	0.373	0.3738	0.3781	0.3475	0.3171	1.4739	1.3516	1.3409	1.4185	1.3084	1.3493
	MSE	2.3216	0.6062	1.1132	0.7662	1.0376	1.0435	0.1347	0.1136	0.1161	0.1271	0.106	0.0825	0.6084	0.9144	0.5783	0.6293	0.2541	0.5157
	Rank	6	1	5	2	3	4	6	3	4	5	2	1	4	6	3	5	1	2
75	mean	1.0259	0.5677	0.6937	0.6425	0.7682	0.7556	0.3513	0.3676	0.3542	0.359	0.3286	0.2989	1.3466	1.2739	1.27	1.3533	1.257	1.2936
	MSE	2.5336	0.3595	0.7026	0.4828	0.855	0.7589	0.0958	0.0948	0.0906	0.1066	0.0849	0.0647	0.2076	0.2139	0.1531	0.3354	0.1307	0.2474
	Rank	6	1	3	2	5	4	5	4	3	6	2	1	3	4	2	6	1	5
100	mean	1.0188	0.5666	0.6751	0.586	0.6812	0.6592	0.3374	0.3427	0.3387	0.3739	0.3288	0.2981	1.3206	1.2718	1.2702	1.3368	1.2619	1.2795
	MSE	2.3718	0.3254	0.6019	0.3945	0.6067	0.5073	0.0815	0.0793	0.0777	0.0986	0.0727	0.0544	0.1341	0.175	0.1211	0.2582	0.1126	0.1555
	Rank	6	1	4	2	5	3	5	4	3	6	2	1	3	5	2	6	1	4
125	mean	0.9428	0.537	0.632	0.5511	0.6509	0.596	0.3233	0.3354	0.3203	0.3789	0.3283	0.313	1.2731	1.2371	1.2252	1.3226	1.2495	1.2723
	MSE	2.047	0.261	0.4807	0.3542	0.5423	0.3955	0.068	0.0701	0.063	0.0945	0.0641	0.0525	0.0854	0.1026	0.0675	0.1793	0.0749	0.1159
	Rank	6	1	4	2	5	3	4	5	2	6	3	1	3	4	1	6	2	5
150	mean	0.9622	0.5202	0.5916	0.5463	0.6285	0.607	0.3221	0.3479	0.3365	0.359	0.3192	0.2962	1.2675	1.2409	1.2349	1.2963	1.2449	1.2539
	MSE	2.0652	0.2455	0.4136	0.308	0.4597	0.3717	0.0646	0.0753	0.0655	0.0833	0.0591	0.0481	0.0666	0.0956	0.0641	0.1217	0.0614	0.0841
	Rank	6	1	4	2	5	3	3	5	4	6	2	1	3	5	2	6	1	4
175	mean	0.9316	0.4817	0.5617	0.5162	0.5996	0.5726	0.3251	0.3571	0.3415	0.3567	0.3226	0.2959	1.2647	1.2423	1.2381	1.2781	1.2336	1.2342
	MSE	1.984	0.1953	0.3582	0.2241	0.414	0.3102	0.0586	0.0702	0.0613	0.0807	0.0567	0.0448	0.0587	0.0774	0.0537	0.0993	0.0492	0.0618
	Rank	6	1	4	2	5	3	3	5	4	6	2	1	3	5	2	6	1	4
200	mean	0.8476	0.4651	0.5225	0.4867	0.5753	0.5593	0.3217	0.3602	0.3426	0.3652	0.3264	0.2996	1.2581	1.2592	1.243	1.2629	1.2217	1.2261
	MSE	1.5794	0.1699	0.2644	0.1962	0.37	0.2911	0.0523	0.0688	0.0571	0.0765	0.0526	0.0417	0.0494	0.0858	0.054	0.0765	0.0364	0.0542
	Rank	6	1	3	2	5	4	2	5	4	6	3	1	2	6	3	5	1	4
225	mean	0.9378	0.4896	0.5735	0.4797	0.5473	0.5325	0.3007	0.3414	0.3185	0.3618	0.3248	0.301	1.2307	1.2219	1.2104	1.254	1.2186	1.217
	MSE	1.854	0.2101	0.3626	0.1889	0.3058	0.2301	0.0488	0.0614	0.0508	0.0744	0.0502	0.0419	0.0303	0.0489	0.0293	0.063	0.0336	0.0429
	Rank	6	2	5	1	4	3	2	5	4	6	3	1	2	5	1	6	3	4
250	mean	0.9027	0.4464	0.5265	0.4683	0.5347	0.5262	0.3129	0.3625	0.3315	0.3686	0.3258	0.3049	1.2319	1.2296	1.2167	1.2653	1.2251	1.2228
	MSE	1.7479	0.1462	0.2629	0.1806	0.2884	0.2352	0.0502	0.0681	0.0531	0.0734	0.0474	0.0398	0.0314	0.0516	0.0324	0.069	0.0324	0.0395
	Rank	6	1	4	2	5	3	3	5	4	6	2	1	1	5	2	6	3	4
Sum of ranks		58	11	39	19	48	35	39	43	36	58	24	10	30	48	20	56	15	41

Table 8. Simulation result for υ=0.3, η=0.8 and λ=0.9

n		υ						η						λ					
		MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE
25	mean	0.9245	0.8384	0.9289	0.9121	0.971	1.0719	1.2518	1.1475	1.1936	1.1029	1.1519	0.8423	1.2155	0.9923	0.9879	1.1867	1.0116	1.149
	MSE	2.3915	1.2225	1.5933	1.4325	1.8047	2.0166	1.8238	1.6064	1.8304	1.4769	1.5831	0.892	1.0029	0.7976	0.5706	1.3565	0.5218	0.8994
	Rank	6	1	3	2	4	5	5	4	6	2	3	1	5	3	2	6	1	4
50	mean	0.9604	0.6672	0.7719	0.7286	0.8356	0.8448	1.0583	1.0065	1.0247	0.9662	0.9383	0.7929	1.0072	0.9154	0.9181	1.0172	0.9596	0.9937
	MSE	2.3022	0.6309	0.9924	0.7463	1.0739	0.9965	1.0174	0.9115	0.9461	0.9037	0.8401	0.5925	0.1668	0.2258	0.1284	0.2458	0.1438	0.1823
	Rank	6	1	3	2	5	4	6	4	5	3	2	1	3	5	1	6	2	4
75	mean	1.0344	0.6155	0.7344	0.5939	0.71	0.734	0.9272	0.9283	0.8995	0.9916	0.9417	0.7986	0.9559	0.8878	0.9039	0.9564	0.9225	0.9528
	MSE	2.2988	0.4009	0.7253	0.3971	0.742	0.7041	0.7092	0.7206	0.689	0.767	0.6849	0.4922	0.0643	0.0901	0.0631	0.1046	0.0615	0.0912
	Rank	6	2	4	1	5	3	4	5	3	6	2	1	3	4	2	6	1	5
100	mean	1.0239	0.5679	0.6568	0.5597	0.645	0.6513	0.8818	0.92	0.9077	0.9679	0.9281	0.7726	0.9464	0.9031	0.9143	0.9426	0.9156	0.9428
	MSE	2.1805	0.3188	0.539	0.315	0.5554	0.4699	0.6133	0.5961	0.5944	0.6941	0.6034	0.3971	0.0411	0.0476	0.0355	0.0762	0.0409	0.0617
	Rank	6	2	4	1	5	3	5	3	2	6	4	1	3	4	1	6	2	5
125	mean	0.9304	0.5101	0.5854	0.5108	0.5961	0.6305	0.8474	0.9268	0.8905	0.9518	0.9267	0.7398	0.9331	0.8927	0.9058	0.9374	0.9185	0.9378
	MSE	1.6536	0.2118	0.3542	0.2315	0.4253	0.361	0.5185	0.5553	0.5051	0.59	0.5425	0.3564	0.0282	0.0373	0.0278	0.0444	0.029	0.0382
	Rank	6	1	3	2	5	4	3	5	2	6	4	1	2	4	1	6	3	5
150	mean	0.8531	0.4815	0.5612	0.4991	0.5586	0.5771	0.866	0.9325	0.8816	0.944	0.8891	0.7709	0.9257	0.8921	0.9056	0.9219	0.91	0.9249
	MSE	1.4607	0.1696	0.3125	0.1963	0.3147	0.2764	0.4868	0.5235	0.4651	0.5876	0.4603	0.3472	0.0246	0.0306	0.0235	0.0387	0.0244	0.0315
	Rank	6	1	4	2	5	3	4	5	3	6	2	1	3	4	1	6	2	5
175	mean	0.8622	0.4888	0.5368	0.4864	0.5505	0.5773	0.8429	0.9236	0.8948	0.9218	0.8551	0.7284	0.9215	0.8892	0.9007	0.9204	0.91	0.9267
	MSE	1.44	0.1828	0.2704	0.1717	0.2774	0.2573	0.4472	0.5234	0.4537	0.5155	0.4247	0.3046	0.0193	0.0268	0.0198	0.0315	0.0193	0.0261
	Rank	6	2	4	1	5	3	3	6	4	5	2	1	2	5	3	6	1	4
200	mean	0.8472	0.4558	0.5123	0.4686	0.5292	0.5576	0.8018	0.9286	0.8793	0.9257	0.8798	0.7284	0.9231	0.8944	0.9051	0.9174	0.9112	0.9213
	MSE	1.3481	0.1318	0.2166	0.1531	0.2537	0.2253	0.3837	0.4903	0.4046	0.4801	0.4158	0.2791	0.0168	0.0241	0.0181	0.025	0.0177	0.0221
	Rank	6	1	3	2	5	4	2	6	3	5	4	1	1	5	3	6	2	4
225	mean	0.8361	0.4589	0.5123	0.4547	0.5098	0.5309	0.789	0.8984	0.8494	0.9191	0.872	0.7359	0.9245	0.9029	0.9132	0.9144	0.9061	0.9236
	MSE	1.2509	0.1365	0.2125	0.1316	0.2174	0.1971	0.3621	0.432	0.3615	0.4715	0.3854	0.256	0.0142	0.0224	0.0159	0.0217	0.0142	0.0203
	Rank	6	2	4	1	5	3	3	5	2	6	4	1	2	6	3	5	1	4
250	mean	0.7486	0.4462	0.5036	0.4495	0.5004	0.5268	0.8115	0.8998	0.848	0.9098	0.8453	0.7392	0.9125	0.9017	0.9061	0.9108	0.9086	0.9214
	MSE	0.9906	0.1166	0.193	0.1216	0.1817	0.1812	0.3362	0.4236	0.3518	0.4384	0.3486	0.2612	0.0123	0.021	0.0144	0.02	0.0133	0.0193
	Rank	6	1	5	2	4	3	2	5	4	6	3	1	1	6	3	5	2	4
Sum of ranks		60	14	37	16	48	35	37	48	34	51	30	10	25	46	20	58	17	44

Table 9. Simulation result for υ=0.3, η=0.8 and λ=1.2

n		υ						η						λ					
		MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE	MLE	LSE	WLSE	CME	ADE	RADE
25	mean	0.7398	0.7671	0.8859	0.8639	0.9257	0.9824	1.2916	1.0949	1.1312	1.1752	1.0843	0.9138	1.9319	1.5094	1.4883	1.7563	1.4803	1.6343
	MSE	1.8402	1.1195	1.6423	1.4569	1.7081	1.7906	1.6636	1.2782	1.3561	1.5342	1.3056	0.8952	3.8452	2.6675	1.8004	3.9311	1.901	2.5496
	Rank	6	1	3	2	4	5	6	2	4	5	3	1	5	4	1	6	2	3
50	mean	0.8689	0.6396	0.7676	0.6738	0.7665	0.7773	1.0948	1.0496	1.0071	1.0689	0.9898	0.8538	1.4808	1.3461	1.3318	1.4766	1.3287	1.3778
	MSE	2.1137	0.6104	1.0423	0.7176	1.0268	0.962	1.0014	0.8992	0.841	0.9734	0.8005	0.5601	0.5653	0.5468	0.4036	1.1082	0.3035	0.6581
	Rank	6	1	5	2	4	3	6	4	3	5	2	1	4	3	2	6	1	5
75	mean	0.9391	0.5622	0.6696	0.6425	0.7585	0.7211	1.0049	0.9974	0.9612	0.9655	0.8749	0.7892	1.3391	1.2581	1.2537	1.3446	1.2562	1.2792
	MSE	2.3192	0.4379	0.7298	0.5027	0.9018	0.6497	0.7062	0.6611	0.641	0.7387	0.5685	0.4464	0.2083	0.217	0.165	0.3432	0.1398	0.1873
	Rank	6	1	4	2	5	3	5	4	3	6	2	1	4	5	2	6	1	3
100	mean	0.931	0.529	0.6192	0.5399	0.6197	0.6195	0.9576	0.9558	0.957	1.049	0.9314	0.8365	1.3134	1.2513	1.2513	1.3333	1.2629	1.2612
	MSE	2.1711	0.274	0.5293	0.3576	0.5248	0.4684	0.606	0.5894	0.5736	0.7223	0.515	0.4104	0.1483	0.1604	0.1205	0.2237	0.1064	0.1203
	Rank	6	1	5	2	4	3	5	4	3	6	2	1	4	5	3	6	1	2
125	mean	0.8991	0.5301	0.6071	0.4987	0.5964	0.5766	0.9011	0.9285	0.9047	1.0475	0.9087	0.8268	1.2707	1.2398	1.2342	1.3089	1.2414	1.2446
	MSE	1.8086	0.2766	0.4673	0.2829	0.498	0.3466	0.4988	0.531	0.4863	0.6596	0.4465	0.3587	0.0768	0.0932	0.0659	0.1689	0.0741	0.0924
	Rank	6	1	4	2	5	3	4	5	3	6	2	1	3	5	1	6	2	4
150	mean	0.9243	0.4747	0.5599	0.5098	0.5845	0.5635	0.8906	0.9831	0.9377	0.997	0.8981	0.8313	1.2732	1.2401	1.2363	1.2915	1.2341	1.2355
	MSE	1.9704	0.2068	0.376	0.2383	0.413	0.3229	0.4622	0.5511	0.4766	0.6423	0.4496	0.3497	0.0751	0.0866	0.063	0.1139	0.0533	0.0784
	Rank	6	1	4	2	5	3	3	5	4	6	2	1	3	5	2	6	1	4
175	mean	0.8583	0.4649	0.5314	0.4523	0.5267	0.4995	0.8898	1.0019	0.9298	1.041	0.9158	0.8732	1.2679	1.258	1.2471	1.2867	1.2384	1.2394
	MSE	1.7393	0.197	0.3054	0.179	0.3062	0.2427	0.3961	0.5447	0.4336	0.6057	0.4016	0.3267	0.0638	0.0961	0.0693	0.1117	0.0562	0.0744
	Rank	6	2	4	1	5	3	2	5	4	6	3	1	2	5	3	6	1	4
200	mean	0.8391	0.4374	0.5116	0.4621	0.5405	0.5226	0.8663	1.0016	0.9254	0.984	0.8647	0.82	1.2507	1.2397	1.2297	1.2582	1.2212	1.2224
	MSE	1.53	0.1515	0.2696	0.1618	0.2756	0.2274	0.3841	0.5129	0.4124	0.5362	0.3725	0.2964	0.0404	0.0577	0.0387	0.0691	0.0387	0.0496
	Rank	6	1	4	2	5	3	3	5	4	6	2	1	3	5	1	6	2	4
225	mean	0.8384	0.4372	0.5099	0.4614	0.5318	0.519	0.8797	1.023	0.9403	1.0116	0.8766	0.8145	1.2509	1.2374	1.2298	1.2704	1.226	1.2262
	MSE	1.5769	0.1487	0.2691	0.1873	0.2846	0.2414	0.3901	0.5584	0.4307	0.549	0.3482	0.2746	0.0409	0.0515	0.0369	0.0771	0.0369	0.0486
	Rank	6	1	4	2	5	3	3	6	4	5	2	1	3	5	2	6	1	4
250	mean	0.8818	0.4408	0.5124	0.4205	0.4885	0.4826	0.8146	0.9664	0.9012	1.0523	0.9329	0.8336	1.2348	1.2337	1.2248	1.2586	1.224	1.2103
	MSE	1.6829	0.1395	0.2549	0.1358	0.2328	0.1729	0.3289	0.464	0.3728	0.5573	0.3715	0.2741	0.0298	0.0459	0.0308	0.0623	0.0319	0.0397
	Rank	6	2	5	1	4	3	2	5	4	6	3	1	1	5	2	6	3	4
Sum of ranks		60	12	42	18	46	32	39	45	36	57	23	10	32	47	19	60	15	37

Table 10. Over all ranks for all set of parameters

		MLE	LSE	WLSE	CME	ADE	RADE
υ=0.3, η=0.3 and λ=0.7	Sum of ranks for υ	58	18	44	12	47	31
	Sum of ranks for η	16	53	40	53	32	15
	Sum of ranks for λ	18	45	26	55	17	49
υ=0.3, η=0.3 and λ=1.2	Sum of ranks for υ	58	11	39	19	48	35
	Sum of ranks for η	39	43	36	58	24	10
	Sum of ranks for λ	30	48	20	56	15	41
υ=0.3, η=0.8 and λ=0.9	Sum of ranks for υ	60	14	37	16	48	35
	Sum of ranks forη	37	48	34	51	30	10
	Sum of ranks forλ	25	46	20	58	17	44
υ=0.3, η=0.8 and λ=1.2	Sum of ranks for υ	60	12	42	18	46	32
	Sum of ranks for η	39	45	36	57	23	10
	Sum of ranks for λ	32	47	19	60	15	37
	Total sum of ranks	472	430	393	513	362	349
	Over all ranks	5	4	3	6	2	1

From the over all Ranks in Table 10 it is preferable to use RADE to estimate the parameters υ, η and λ.

8. Applications

This section analyzes two real-world data set from renewable energy sources to demonstrate the applicability and flexibility of the ATKE distribution. The goodness-of-fit statistics for these distributions and other competitive distributions were utilized to compare our novel models with the current competing models. Furthermore, MLEs of parameters and standard errors (SEs) are given in Tables 11 and 12 for the proposed data sets. The first dataset shows the proportion of global CO2 emissions per person for 211 nations in 2020. The following data, which were previously utilized in [60], are provided: 0.18, 1.88, 0.58, 3.53, 20.32, 5.39, 7.41, 0.11, 0.68, 2.09, 0.71, 0.26, 0.26, 0.21, 3.8, 0.73, 3.78, 0.99, 0.31, 2.16, 1.76, 5.01, 11.47, 6.53, 0.94, 3.37, 1.93, 6.08, 7.69, 0.67, 5, 0.04, 15.37, 0.56, 4.85, 14, 6.75, 4.66, 9.06, 1.68, 2.62, 2.56, 0.36, 15.52, 1.36, 0.57, 1.75, 0.08, 6.04, 1.75, 3.32, 8.6, 2.5, 2.56, 6.26, 0.92, 0.03, 7.62, 17.97, 0.59, 1.99, 1.53, 1.06, 0.4, 5.63, 5.24, 8.42, 6.94, 0.43, 4.89, 7.09, 3.47, 13.06, 0.64, 8.15, 1.02, 0.13, 3.99, 12.12, 0.43, 5.07, 2.5, 1.14, 0.04, 5.94, 1.06, 4.47, 0.07, 4.99, 1.93, 8.23, 0.38, 1.24, 5.02, 1.47, 6.73, 0.51, 30.45, 0.36, 20.55, 12.17, 0.77, 0.62, 26.98, 2.36, 3.96, 2.38, 4.24, 2.4, 1.56, 3.79, 2.44, 2.98, 7.32, 0.07, 4.65, 3.43, 6.51, 0.2, 3.61, 23.22, 12.49, 0.99, 15.19, 3.83, 0.26, 7.05, 2.77, 14.24, 4.25, 4.94, 2.51, 0.05, 0.98, 0.15, 3.72, 1.55, 7.62, 2.5, 5.07, 0.06, 0.3, 1.24, 6.98, 5.23, 1.55, 10.81, 2.2, 1.77, 0.11, 7.92, 6.4, 2.81, 11.66, 6.03, 2.95, 1.74, 0.56, 1.36, 0.61, 0.74, 0.17, 3.7, 0.99, 0.11, 8.87, 0.21, 2.77, 0.2, 4.52, 25.37, 14.2, 5.24, 20.83, 1.28, 3.69, 0.82, 3.59, 1.78, 8.06, 5.38, 3.73, 8.22, 7.23, 2.5, 3.68, 1.77, 0.33, 0.13, 0.55, 4.52, 0.19, 1.06, 2.61, 4.14, 1.58, 37.02, 8.74, 4.4, 4.61, 7.88, 0.51, 1.75, 10.03, 3.72, 1.94, 0.3, 3.13, 0.26, 7.78, 7.38.

The second set of data shows the proportion of 75 nations' primary energy consumption in 2019 that was derived from renewable technology. The following information, which [60] previously used, is available: 0.031, 0.6618, 0.0402, 0.0799, 0.0618, 0.0325, 0.0064, 0.2285, 0.0654, 0.2539, 0.1064, 0.0618, 0.0871, 0.0213, 0.0518, 0.092, 0.1266, 0.7908, 0.1629, 0.1141, 0.0077, 0.0251, 0.1649, 0.3016, 0.2499, 0.0056, 0.3064, 0.2615, 0.3039, 0.0694, 0.0931, 0.073, 0.0888, 0.0456, 0.0254, 0.0311, 0.1445, 0.0248, 0.0609, 0.4502, 0.354, 0.0587, 0.1092, 0.1356, 0.1697, 0.0709, 0.0778, 0.0221, 0.0406, 0.1071, 0.0027, 0.2445, 0.0722, 0.159, 0.2154, 0.1173, 0.0393, 0.023, 0.4224, 0.0101, 0.0834, 0.1522, 0.1818, 0.1211, 0.1847, 0.1748, 0.1405, 0.2733, 0.337, 0.1545, 0.0601, 0.0857, 0.1054, 0.2764, 0.1574.

The real data sets are utilized to assess the goodness of fit of the ATKE distribution. The suggested model is compablack with KE, NMWe, TMWe, EExWe, BWe, LBTLoWe, MWe, IPLEx and HLMKE models. For each model, Tables 11 and 12 provide the MLEs and SEs of the model parameters.

Table 11. The MLEs and their SEs for first data set

Model	υ	St(υ)	η	St(η)	λ	St(λ)	α	St(α)	β	St(β)
ATKE	2.9483	3.6453	0.0470	0.0635	0.8369	0.0616				
KE	30.7817	7.2761	0.0037	0.0007	0.8240	0.0465				
NMWe	0.2552	0.1721	0.0513	0.1701	0.8580	0.1309	0.0385	0.1317	0.8553	0.4374
TMWe	0.5035	2.6679	0.9402	0.2225	0.8231	2.6489	0.0516	0.4631		
EExWe	1.1839	0.4001	1.7325	NaN	0.7572	0.1467	0.5952	NaN		
BWe	0.1240	0.2969	0.7404	0.175	1.2195	0.4791	1.9909	3.7550		
LBTLoWe	0.4517	1.8415	0.5919	0.0697	0.6738	0.1961				
IPLE	188.9536	32.8129	0.0046	0.0008	0.6959	0.0006				
MWe	0.6570	0.1335	1.1441	0.2889	0.0917	0.0200				
HLMKE	0.6573	0.1332	1.1436	0.2881	0.0917	0.0200				

Table 12. The MLEs and their SEs for second data set

Model	υ	St(υ)	η	St(η)	λ	St(λ)	α	St(α)	β	St(β)
ATKE	0.7515	2.2877	8.7964	25.1302	1.2866	0.3441				
TMWe	27.3086	117.8586	1.0232	0.0870	33.0604	117.8933	0.6999	0.4270		
NMWe	705.4868	4640.1026	15.2193	5.9091	27.8863	30.2041	1.2482	0.6310	1.3040	0.1512
KE	0.2849	NaN	26.3705	NaN	1.3647	NaN				
IPLE	199.8841	47.3047	0.0059	0.0014	0.6836	0.0022				
EExWe	1.6804	0.9419	0.0066	0.0024	0.8317	0.2432	0.1115	0.1260		
BWe	10.4545	42.5397	0.8267	0.4213	1.6940	1.2875	1.0451	4.601		
LBTLoWe	1.9862	4.9948	0.7212	0.2165	7.1802	1.0359				
MWe	0.6933	0.221	1.5144	0.6176	3.7257	1.2510				
HLMKE	0.6933	0.2211	1.5145	0.6180	3.7260	1.2518				

Several criteria are taken into consideration to evaluate the distribution models. These criteria include the Akaike information criterion (H_1), the Kolmogorov-Smirnov test (H_2), the p-value (H_3) test, the Cramer-Von-Mises test (H_4), and the Anderson-Darling test (H_5). This is in contrast to the broader dispersion, which is related to lower values of H_1, H_2, H_4and H_5, as well as the maximum value of H_3. The values for the proposed criteria measures are given in Tables 13 and 14.

The three-parameter ATKE distribution has demonstrated a higher level of goodness of fit when compablack to other models. Among the distributions that are being taken into account for this study, this particular distribution has the most incblackible value of H_3 and the lowest values of H_1, H_2,

Table 13. Measures of fitting for data set 1.

Name	H_1	H_2	H_3	H_4	H_5
ATKE	1057.4290	0.0398	0.8917	0.0535	0.3876
KE	1057.7220	0.0427	0.8373	0.0695	0.4531
NMWe	1061.1850	0.0428	0.8333	0.0646	0.4237
TMWe	1059.0120	0.0436	0.8179	0.0655	0.4240
EExWe	1059.0710	0.0448	0.7919	0.0696	0.4419
BWe	1059.0530	0.0450	0.7858	0.0706	0.4456
LBTLoWe	1059.0060	0.0627	0.3776	0.1454	0.8076
IPLE	1082.2750	0.0723	0.2204	0.4640	2.6597
MWe	1071.4420	0.0997	0.0302	0.0991	0.8156
HLMKE	1071.4420	0.0997	0.0301	0.0991	0.8152

Table 14. Measures of fitting for data set 2.

Model	H_1	H_2	H_3	H_4	H_5
ATKE	-137.3060	0.0422	0.9993	0.0184	0.1445
TMWe	-135.4829	0.0481	0.9951	0.0203	0.1519
NMWe	-136.0730	0.0515	0.9887	0.0192	0.1513
KE	-136.9961	0.052	0.9873	0.0235	0.176
IPLE	-131.9011	0.0531	0.9840	0.0767	0.5326
EExWe	-135.2215	0.0555	0.9752	0.0212	0.1628
BWe	-135.2210	0.0556	0.9745	0.0213	0.1630
LBTLoWe	-137.1723	0.0557	0.9742	0.0217	0.1680
MWe	-130.9251	0.1100	0.3238	0.0691	0.4877
HLMKE	-130.9251	0.1101	0.3237	0.0691	0.4877

H_4 and H_5. Additionally, this distribution has the lowest value of H_5. Besides that, Figures 3-6 show the estimated PDF, CDF, SF, and probability-probability (PP) plots for the competitive model that was used with the given data sets.

As a result of the above tables and figures, we have arrived at the conclusion that the ATKE model provides the best overall fit, and as a result, it is possible to choose it as the model that is most suitable for describing the data sets.

9. Conclusions

This study introduces the ATKE distribution as an extension of the KE distribution. The ATKE distribution is a combination of the arctan-X family of distributions and the KE distribution. Compablack with the traditional KE distribution, the ATKE distribution is flexible and may represent a variety of hazard rate shapes. The densities exhibit many asymmetric and unimodal shapes. The hazard rate functions show the numerous varieties of declining, rising, increasing-constant, and inverted j-shaped forms. Some statistical features of the developed model are conducted. To estimate the parameters of the ATKE distribution, six widely used statistical approaches, including ML, WLS, AD, LS, CVM

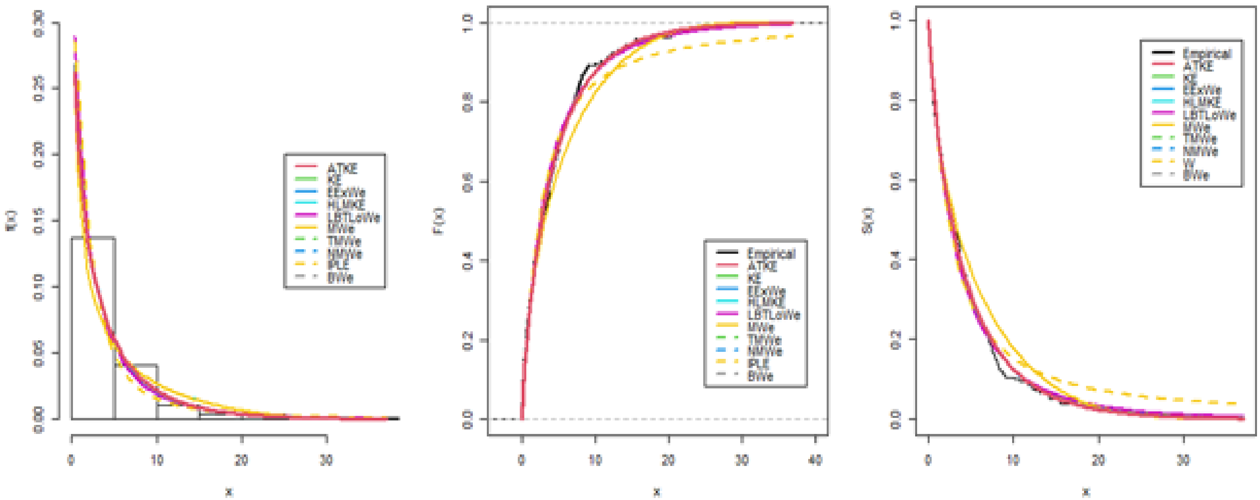

Figure 3. Estimated PDF, CDF and SF plots for data set 1

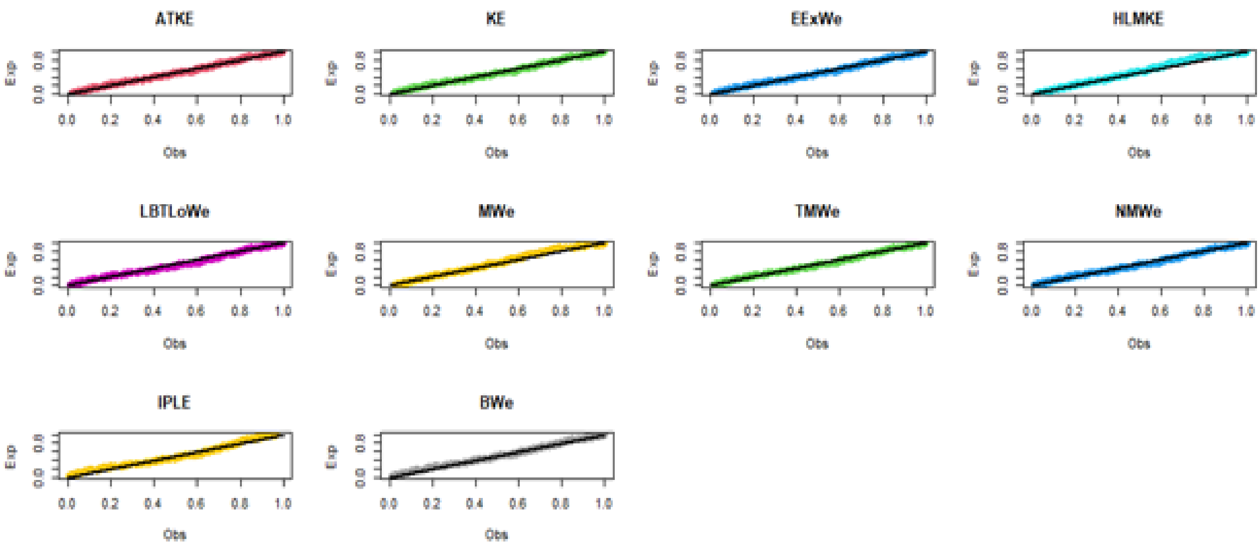

Figure 4. The PP plots of the fitted model for data set 1

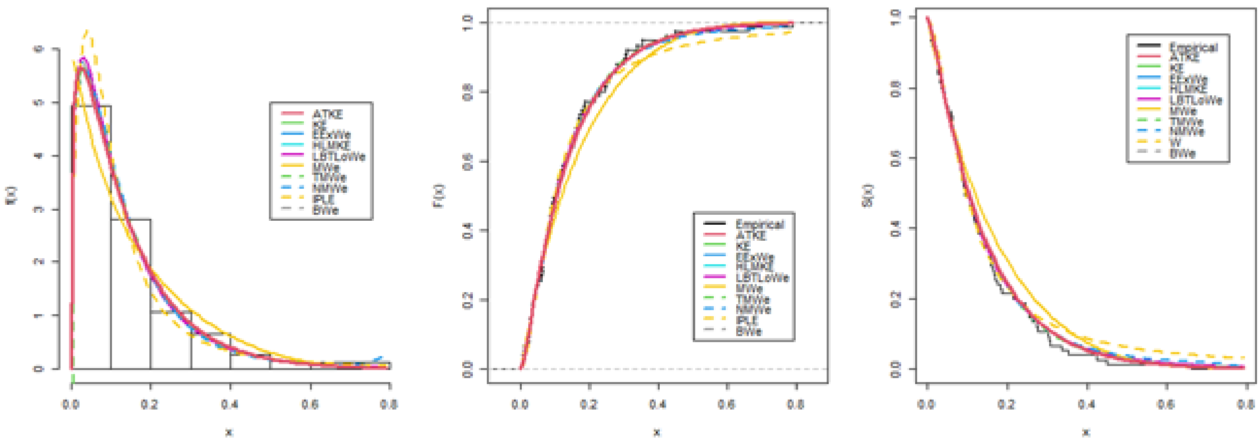

Figure 5. Estimated PDF, CDF and SF plots for data set 2

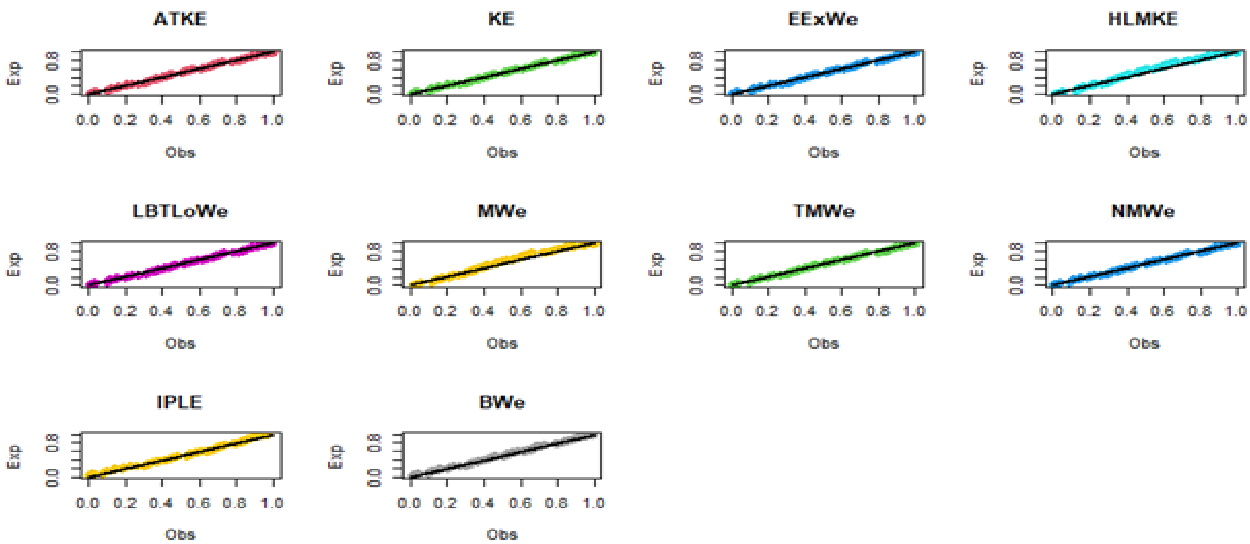

Figure 6. The PP plots of the fitted model for data set 2

RAD are used. Through a comprehensive simulation analysis, we were able to demonstrate the efficacy of different estimates. Furthermore, the recommended model's adaptability was tested on a dataset of renewable energy sources, showing that it might be able to match the data better than some other competing models with two, three, and four parameters.

Conflict of Interest: The authors declare no competing interests.

Data Availability: Any data that supports the findings of this study is included in the article.

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